

CSM – 52/17
Mathematics
Paper – I

Time : 3 hours

Full Marks : 300

The figures in the right-hand margin indicate marks.

Candidates should attempt Q. No. 1 from Section – A and Q. No. 5 from Section – B which are compulsory and three of the remaining questions selecting at least one from each Section.

SECTION – A

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1. Answer any five of the following :

- (a) Prove that every subgroup of a cyclic group is itself a cyclic group. 12
- (b) Let $S = \{(x, y, z) / x + y + z = 0\}$, x, y, z being real. Prove that S is a subspace of \mathbb{R}^3 . Find a basis of S . 12

- (c) State Cayley-Hamilton theorem and use it to calculate the inverse of the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 2 & 3 & 2 \end{pmatrix}$$

12

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- (d) A variable plane is at a constant distance p from the origin O and meets the axes at A , B and C . Show that the locus of the centroid of the tetrahedron $OABC$ is

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{16}{p^2}$$

12

- (e) Show that the triangle formed by the straight lines $ax^2 + 2hxy + by^2 = 0$ and the straight line $\ell x + my = 1$ is right angled if

$$(a + b)(a\ell^2 + 2h\ell m + bm^2) = 0$$

12

- (f) Prove that a finite integral domain is a field.

12

2. (a) If p be a prime and p is not a divisor of a , then $a^{p-1} \equiv 1 \pmod{p}$.

15

- (b) If H is a cyclic normal subgroup of a group G , then show that every subgroup of H is normal in G .

15

(c) If Z is the set of integers then show that $Z[\sqrt{-3}] = \{a + b\sqrt{-3} : a, b \in Z\}$ is not a unique factorization domain. 15

(d) Let F be a field and let $p(x)$ be an irreducible polynomial over F . Let $\langle p(x) \rangle$ be the ideal generated by $p(x)$. Prove that $\langle p(x) \rangle$ is a maximal ideal. 15

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3. (a) Find a basis for \mathbb{R}^3 that contains the vectors $(1, 2, 0)$ and $(1, 3, 1)$. 15

(b) Let $T : M_{2,1} \rightarrow M_{2,3}$ be a linear transformation defined by (with usual notations)

$$T \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 3 \\ 4 & 1 & 5 \end{pmatrix}, T \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}. \text{ Find}$$

$$T \begin{pmatrix} x \\ y \end{pmatrix}. \quad 15$$

(c) Obtain the normal form under congruence and find the rank and signature of the

$$\text{symmetric matrix } \begin{pmatrix} 2 & 4 & 3 \\ 4 & 6 & 3 \\ 3 & 3 & 1 \end{pmatrix}. \quad 15$$

- (d) Show that the eigen values of a real symmetric matrix are all real. 15

4. (a) If a point lies on the ellipse $\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 1$,

prove that its polar with respect to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ touches the ellipse}$$

$$\frac{a'^2}{a^4}x^2 + \frac{b'^2}{b^4}y^2 = 1. \quad 15$$

- (b) Prove that the straight line $\ell x + my + n = 0$ touches the parabola $y^2 - 4ax + 4ab = 0$ if $\ell^2 b + \ell n = am^2$. 15

- (c) Find the equation of the plane passing through the straight line $3x + y + 2z - 7 = 0 = x + y - z + 4$ and perpendicular to the plane $2x + y + z = 5$. 15

- (d) Show that the angle between the lines of section of the plane $3x + y + 5z = 0$ and the

cone $6yz - 2zx + 5xy = 0$ is $\cos^{-1} \left(\frac{1}{6} \right)$. 15

SECTION – B

5. Answer any five of the following :

(a) Prove that a monotone increasing sequence, if bounded above, is convergent and the sequence converges to the upper bound. 12

(b) Find Laurent series for : 12

(i) $\frac{e^{2z}}{(z-1)^3}$ about $z = 1$

(ii) $\frac{1}{z^2(z-3)^2}$ about $z = 3$

(c) By means of contour integration, evaluate

$$\int_0^{\infty} \frac{(\log_e u)^2}{u^2 + 1} du. \quad 12$$

(d) Examine the convergence of the series

$$1 + \frac{x}{1!} + \frac{2^2 x^2}{2!} + \frac{3^3 x^3}{3!} + \dots, \quad x > 0. \quad 12$$

(e) An area bounded by a quadrant of a circle of radius a and the tangents at its extremities revolves about one of the tangent. Find the volume so generated. 12

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- (f) Calculate the curvature at the point u of the curve given by the parametric equations $x = a(3u - u^3)$, $y = 3au^2$, $z = a(3u + u^2)$. 12

6. (a) Let $f(x) = \begin{cases} 0, & x \text{ is irrational} \\ 1, & x \text{ is rational} \end{cases}$

Show that f is not Riemann-integrable on $[a, b]$. 15

- (b) If $\lim_{n \rightarrow \infty} a_n = \ell$, then prove that

$$\lim_{n \rightarrow \infty} \frac{a_1 + a_2 + \dots + a_n}{n} = \ell. \quad 15$$

- (c) If $u + v = \frac{2 \sin 2x}{e^{2y} + e^{-2y} - 2 \cos 2x}$ and $f(z) = u + iv$ is an analytic function of $z = x + iy$, find $f(z)$ in terms of z . 15

- (d) Use the method of contour integration to

prove that $\int_0^{2\pi} \frac{d\theta}{1 + a^2 - 2a \cos \theta} = \frac{2\pi}{1 - a^2}$, $0 < a < 1$. 15

7. (a) Express $\int_0^1 x^m (1 - x^n)^p dx$ in terms of

Gamma function and hence evaluate the

integral $\int_0^1 x^6 \sqrt{1 - x^2} dx$. 15

(b) Discuss the applicability of Rolle's theorem

$$\text{on } f(x) = e^{-x} (\cos x - \sin x) \text{ in } \left[\frac{\pi}{4}, \frac{5\pi}{4} \right].$$

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(c) Find the area of the region bounded by the upper half of the circle $x^2 + y^2 = 25$, the x-axis and the ordinates $x = -3$ and $x = 4$.

15

(d) Find the surface of a sphere generated by the circle $x^2 + y^2 = a^2$ about x-axis.

15

8. (a) Evaluate $\vec{\nabla} \cdot \left[r \vec{\nabla} \left(\frac{1}{r^3} \right) \right]$, where

$$r = \sqrt{x^2 + y^2 + z^2}. \quad 15$$

(b) Prove that if $\int_{P_1}^{P_2} \vec{F} \cdot d\vec{r}$ is independent of the

path joining any two points P_1 and P_2 in a given region, then $\oint \vec{F} \cdot d\vec{r} = 0$ for all closed paths in the region and conversely.

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(c) Prove that $\iiint_V \nabla \phi \, dv = \iint_S \phi \vec{n} \, ds$, where \vec{n} is unit normal vector to the surface S. 15

(d) Using Green's theorem in the plane, show that the area bounded by a simple closed curve C is given by $\frac{1}{2} \oint_C (x \, dy - y \, dx)$. Hence, find the area of the ellipse $x = a \cos \theta$, $y = b \sin \theta$. 15

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