

**CSM – 52 / 15**

**Mathematics**

**Paper – I**

*Time : 3 hours*

*Full Marks : 300*

*The figures in the right-hand margin indicate marks.*

*Candidates should attempt Q. No. 1 from Section – A and Q. No. 5 from Section – B which are compulsory and **three** of the remaining questions, selecting at least **one** from each Section.*

**Section – A**

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1. Answer any **three** of the following :

- (a) Prove that the order of a subgroup  $H$  of a finite group  $G$  divides the order of  $G$ . 20
- (b) In a vector space  $V$  of dimension  $n$ , prove that a subset of  $n$  vectors is linearly independent if and only if it spans  $V$ . 20

- (c) Find a matrix of order 3 whose eigenvalues are 1, 2, 3 and the corresponding eigen-

vectors are  $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ ,  $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$  and  $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$  respectively.

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- (d) Show that the equation :

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$$\frac{a}{y-z} + \frac{b}{z-x} + \frac{c}{x-y} = 0$$

represents a pair of planes.

2. (a) If  $x$  and  $y$  are relatively prime and not both zero, show that there are integers  $m$  and  $n$  such that the greatest common divisor of  $x$  and  $y$  equals  $mx + ny$ . 15
- (b) If  $G$  is a finite group and  $p$  is a prime number such that  $p^\alpha$  divides  $o(G)$ , show that there is a subgroup of  $G$  of order  $p^\alpha$ . 15
- (c) If  $G$  is an Abelian group, show that  $(a \cdot b)^n = a^n \cdot b^n$  for all integers  $n$  and for all  $a, b, \in G$ . Show that converse is true if  $(a \cdot b)^n = a^n \cdot b^n$  for three consecutive integers  $n$  and for all  $a, b \in G$ . 15

- (d) Let  $R$  be a commutative ring with unit element and  $M$  is an ideal of  $R$ . Show that  $M$  is a maximal ideal if and only if  $R/M$  is a field. **Techofworld.In** 15
3. (a) Let  $V$  be a vector space of dimension  $n$  over a field  $F$ . Show that the transformation that maps a vector  $v \in V$  to the corresponding coordinate vector with respect to a fixed basis of  $V$  is a linear transformation from  $V$  to  $F^n$ . 15
- (b) If  $V$  is a finite-dimensional vector space and if  $W$  is a subspace of  $V$ , show that  $W$  is also finite-dimensional and  $\dim W \leq \dim V$  and  $\dim(V/W) = \dim V - \dim W$ . Show that  $\dim W = \dim V$  if and only if  $W = V$ . 15
- (c) Show that the eigenvalues of a Hermitian matrix are real. Deduce that the eigenvalues of a symmetric matrix with real entries are real. 15
- (d) Let  $A$  be a matrix of order  $n$  and let  $\text{adj } A$  denote the adjoint matrix of  $A$ . Show that : 15
- (i)  $A$  is invertible if and only if  $A^T A$  is invertible.

(ii) A is invertible if and only if  $\text{adj } A$  is invertible. **Techofworld.In**

(iii) The determinant of  $\text{adj } A$  equals  $|A|^{n-1}$ .

4. (a) Tangents are drawn from the point  $P(h, k)$  to the circle  $x^2 + y^2 = a^2$ . These tangents touch the circle at  $A, B$ . Find the area of the triangle  $ABP$ . 15

(b) Find the value of  $\lambda$  so that the equation  $x^2 - \lambda xy + 2y^2 + 3x - 5y + 2 = 0$  may represent a pair of straight lines. Find the lines. 15

(c) Let a point move so that the sum of the squares of its distances from the six faces of a cube is constant. Prove that its locus is a sphere. 15

(d) Find the condition so that the plane  $lx + my + nz = p$  may touch the central conicoid  $ax^2 + by^2 + cz^2 = 1$ . 15

### Section – B

5. Answer any **three** of the following :

(a) Show that the interval  $[0, 1]$  is uncountable.

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(b) If  $f(z) = u + iv$  is analytic and  $u^4 + uv = 2016$ , show that  $f$  is a constant. 20

(c) Evaluate the integral :

$$\oint_C \left( \bar{z} + \frac{1}{z} \right)^3 dz$$

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where  $C$  is the circle  $|z| = 1$  in the anticlockwise direction. 20

(d) Show that the series  $\sum_{n=1}^{\infty} n^{-p}$  converges for  $p > 1$  and diverges for  $0 < p \leq 1$ . 20

6. (a) Show that a monotonically increasing function on an interval  $[0, 1]$  is Riemann integrable. 15

(b) Suppose that the sequence  $\langle f_n \rangle$  of Riemann integrable functions of  $[0, 1]$  converges to a function  $f$ . Show that the function  $f$  is Riemann integrable and

$$\int_0^1 f(x) dx = \lim_{n \rightarrow \infty} \int_0^1 f_n(x) dx. \quad 15$$

(c) Find the Laurent series expansion of the function  $f(z) = 1/(z^2 - 3z + 2)$  valid in : 15

(i)  $1 < |z| < 2$

(ii)  $|z - 1| < 1$

(iii)  $1 < |z - 1| < 2$

(iv)  $|z - 2| < 1$

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(d) Evaluate the integral

$$\oint_C \frac{dz}{z^4 - 3z^2 - 4}$$

where C is the circle  $|z - 1| = 2$  in the anticlockwise direction. 15

7. (a) If  $\alpha > 1$ , show that the inequality  $(1 + x)^\alpha \geq 1 + \alpha x$  holds for all  $x > -1$ . Discuss the case of equality. 15

(b) Express the integral  $\int_0^1 x^m (\log x)^n dx$  when  $m, n > 0$  in terms of gamma function. 15

(c) Find the area of the surface of the solid generated by revolving astroid  $x = a \cos^3 t$ ,  $y = a \sin^3 t$  about the axis of x. 15

(d) Evaluate the triple integral

$$\iiint_G xy^2z^3 dV$$

over the rectangular box  $G$  defined by the inequalities  $-1 \leq x \leq 2$ ,  $0 \leq y \leq 3$  and  $0 \leq z \leq 2$ .

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8. (a) Show that  $\text{div}(\text{curl } F) = 0$  and  $\text{curl}(\nabla\phi) = 0$ .  
Verify these results directly when  $F(x, y, z) = x^2y\hat{i} + 2y^3z\hat{j} + 3zk\hat{k}$  and  $\phi(x, y, z) = x^2y + y^2z + z^2x$ . 15
- (b) Find the work done by the force-field  $F(x, y, z) = (e^x - y^3)\hat{i} + (\cos y + x^3)\hat{j}$  on a particle that travels once around the unit circle  $x^2 + y^2 = 1$  in the clockwise direction. 15
- (c) Using Gauss divergence theorem, find the outward flux of the vector field  $F(x, y, z) = x^3\hat{i} + y^3\hat{j} + z^3\hat{k}$  across the surface of the region that is enclosed by the circular cylinder  $x^2 + y^2 = 9$  and the planes  $z = 0$  and  $z = 2$ . 15
- (d) Let  $F(x, y, z) = 2z\hat{i} + 3x\hat{j} + 5y\hat{k}$ ,  $\sigma$  be the portion of the paraboloid  $z = 4 - x^2 - y^2$  for which  $z \geq 0$  with upward orientation and  $C$  be the positively oriented circle  $x^2 + y^2 = 4$

that forms the boundary of  $\sigma$  in the  $xy$ -plane.

Compute  $\oint_C \mathbf{F} \cdot d\mathbf{r}$  and  $\iint_{\sigma} (\text{curl } \mathbf{F}) \cdot n dS$ . Does

this verify Stokes' theorem ?

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